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New Solitary Wave Solutions to the KdV-Burgers Equation

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Based on the analysis on the features of the Burgers equation and KdV equation as well as KdV-Burgers equation, a superposition method is proposed to construct the solitary wave solutions of the KdV-Burgers equation from those of the Burgers equation and KdV equation, and then by using it we obtain many solitary wave solutions to the KdV-Burgers equation, some of which are new ones.

KEY WORDS: KdV-Burgers equation; superposition approach; solitary wave solution.
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1. INTRODUCTION

At present, more and more problems in branches of modern mathematical physics and other interdisciplinary science are described in terms of suitable nonlinear models. Therefore, it is still a very important and essential task to search for the explicit and exact solutions (in particular, the solitary wave solutions) to nonlinear evolution equations in nonlinear science. In recent years, a large number of simple and direct approaches have been put forward to construct the explicit and exact solutions (especially the solitary wave solutions) of nonlinear evolution equations. Among these are the homogeneous balance method (Wang, 1995, 1996), the tanh-function method (Malfliet, 1992; Parkes and Duffy, 1997; Zhang, 1999), the sech-function method (Duffy and Parkes, 1996), the sine-cosine method (Yan and Zhang, 1999), the trial function method (Otwinowski *et al.*, 1998; Liu *et al.*, 2001), the Jacobi elliptic function expansion method (Fu *et al.*, 2001; Liu *et al.*, 2001; Parkes *et al.*, 2002), the mapping method (Peng, 2003) and so on. However, not all the above approaches are universally workable for solving all kinds of nonlinear evolution equations directly. As a consequence, it

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is still a very significant work to go on looking for various powerful and efficient approaches to solve nonlinear evolution equations.

In the present paper, by analyzing carefully the features of the Burgers equation and KdV equation as well as KdV-Burgers equation, we present a superposition method which is commonly believed to simply work for linear partial differential equations to construct the explicit and exact solitary wave solutions of the KdV-Burgers equation from those of the Burgers equation and KdV equation, and then by means of it we derive a number of explicit and exact solitary wave solutions of the KdV-Burgers equation, among which are some new ones.

2. SOLUTIONS TO THE KDV-BURGERS EQUATION

It is well known that the celebrated Burgers equation and KdV equation as well as KdV-Burgers equation are of the following general form, respectively

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{3}$$

where α and β are arbitrary constants with $\alpha\beta \neq 0$.

The above three equations are probably the most popular nonlinear evolution equations of physical interest, which not only stem from realistic physical phenomena, but can also be widely applied to a lot of physically significant fields such as plasma physics, fluid dynamics, crystal lattice theory, nonlinear circuit theory and astrophysics. In the present paper, it was of interest to us to study the explicit and exact solitary wave solutions to the KdV-Burgers equation which arises in many different physical contexts as a nonlinear model equation incorporating the effects of dispersion and dissipation as well as nonlinearity and which was used by Liu and Liu (1991), to model the inverse energy cascade and intermittent turbulence in which a dispersion effect is considered by means of the superposition method. To begin with, let us analyze carefully the features of the Burgers equation and KdV equation as well as KdV-Burgers equation. From Eqs. (1) to (3), it is not hard to see that they are all of the same nonlinear term $u\frac{\partial u}{\partial x}$ and of distinct linear terms and that the linear terms of Eq. (3) $(-\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3})$ are exactly equal to the superposition of those of Eq. (1) and Eq. (2). For this reason, we may construct the solutions of Eq. (3) by the linear superposition of those to Eq. (1) and Eq. (2), in other words, we may assume that Eq. (3) has the following ansatz solution:

$$u = au_B + bu_K \tag{4}$$

where a and b are constants to be determined later, and u_B is the solution of Eq. (1), and u_K the solution of Eq. (2). We call Eq. (4) as the superposition formula to find the solutions of the KdV-Burgers equation (3). In what follows, let us seek the solitary wave solutions to Eq. (3) by Eq. (4).

Form reference (Xie and Tang, 2004), we are told that the Burgers equation (1) has the following kink-type solitary wave solution:

$$u_B = -\alpha k \left(1 + \tanh \frac{k}{2}\xi\right) \tag{5}$$

and that the KdV equation (2) has the following bell-type solitary wave solution:

$$u_K = 12\beta k^2 \operatorname{sech}^2 k\xi \tag{6}$$

where

$$\xi = x - ct \tag{7}$$

Here k and c are the wave number and the wave velocity, respectively.

According to Eqs. (5)–(7) and making use of Eq. (4), we suppose that Eq. (3) has the following ansatz solution:

$$u = a(1 + \tanh k\xi) + b \operatorname{sech}^2 k\xi \tag{8}$$

Inserting Eq. (8) into Eq. (3), and with the aid of the computerized symbolic computation of the powerful Mathematica, we can obtain

$$\begin{aligned} &a^2k + abk + 2bk^2\alpha - 2ak^3\beta - ack \\ &\quad + (a^2k - 2abk - 2b^2k + 2ak^2\alpha + 16bk^3\beta + 2bck) \tanh k\xi \\ &\quad + (-a^2k - 4abk - 8bk^2\alpha + 8ak^3\beta + ack) \tanh^2 k\xi \\ &\quad + (-a^2k + 2abk + 4b^2k - 2ak^2\alpha - 40bk^3\beta - 2bck) \tanh^3 k\xi \\ &\quad + (3abk + 6bk^2\alpha - 6ak^3\beta) \tanh^4 k\xi + (-2b^2k + 24bk^3\beta) \tanh^5 k\xi = 0 \end{aligned} \tag{9}$$

Setting each of the coefficients of $\tanh^j k\xi$ $(j = 0, 1, 2, \ldots, 5)$ in Eq. (9) to zero gives rise to a set of over-determined algebraic equations with regard to the unknown variables a and b as follows:

$$a^2k + abk + 2bk^2\alpha - 2ak^3\beta - ack = 0 \tag{10}$$

$$a^2k - 2abk - 2b^2k + 2ak^2\alpha + 16bk^3\beta + 2bck = 0 \tag{11}$$

$$-a^2k - 4abk - 8bk^2\alpha + 8ak^3\beta + ack = 0 \tag{12}$$

$$-a^2k + 2abk + 4b^2k - 2ak^2\alpha - 40bk^3\beta - 2bck = 0 \tag{13}$$

$$3abk + 6bk^2\alpha - 6ak^3\beta = 0 \tag{14}$$

$$-2b^2k + 24bk^3\beta = 0 \tag{15}$$

Solving the above system of algebraic equations by means of Mathematica, we have the following results:
Case 1

$$a = \frac{6\alpha^2}{25\beta}, \qquad b = \frac{3\alpha^2}{25\beta}, \qquad k = -\frac{\alpha}{10\beta}, \qquad c = \frac{6\alpha^2}{25\beta} \tag{16}$$

Substituting Eq. (16) into Eq. (8) and utilizing Eq. (7), we obtain the following solitary wave solution to the KdV-Burgers equation (3):

$$u_1 = \frac{12\alpha^2}{25\beta} - \frac{3\alpha^2}{25\beta}\left[1 + \tanh\frac{\alpha}{10\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{17}$$

Case 2

$$a = -\frac{6\alpha^2}{25\beta}, \qquad b = \frac{3\alpha^2}{25\beta}, \qquad k = \frac{\alpha}{10\beta}, \qquad c = -\frac{6\alpha^2}{25\beta} \tag{18}$$

Plugging Eq. (18) into Eq. (8) and using Eq. (7), we get the following solitary wave solution to the KdV-Burgers equation (3):

$$u_2 = -\frac{3\alpha^2}{25\beta}\left[1 + \tanh\frac{\alpha}{10\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{19}$$

Making use of the following two equalities

$$\tanh\frac{x}{2} = \frac{\sinh x}{1 + \cosh x} \tag{20}$$

and

$$\sinh^2 x = \cosh^2 x - 1 \tag{21}$$

Eq. (17) and (19) can be rewritten respectively as,

$$u_3 = \frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\frac{\sinh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)}{1 + \cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)} + \frac{6\alpha^2}{25\beta}\frac{1}{1 + \cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)} \tag{22}$$

$$u_4 = -\frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta} \frac{\sinh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)}{1 + \cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)} + \frac{6\alpha^2}{25\beta} \frac{1}{1 + \cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)} \tag{23}$$

Similarly, making use of the identity (21) and the following identity

$$\tanh\frac{x}{2} = \frac{\cosh x - 1}{\sinh x} \tag{24}$$

Equation (17) and (19) can be reduced respectively to,

$$u_5 = \frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta} \frac{\cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) - 1}{\sinh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)} + \frac{6\alpha^2}{25\beta} \frac{1}{1 + \cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)} \tag{25}$$

$$u_6 = -\frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta} \frac{\cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) - 1}{\sinh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)} + \frac{6\alpha^2}{25\beta} \frac{1}{1 + \cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)} \tag{26}$$

In addition, from the theorem given in reference (Liu, 2003), we are told that if a given NLLEE has the kink-type solution $u(x,t) = P[\tanh k(x - ct)]$, then it has certainly the kink-bell-type solution $u(x,t) = P[\tanh 2k(x - ct) \pm \text{isech}\, 2k(x - ct)]$. So according to the above theorem and Eqs. (17) and (19), the KdV-Burgers equation (3) admits also the following kink-bell-type solutions:

$$u_{7,8} = \frac{12\alpha^2}{25\beta} - \frac{3\alpha^2}{25\beta}\left[1 + \tanh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) \pm i\,\text{sech}\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{27}$$

$$u_{9,10} = -\frac{3\alpha^2}{25\beta}\left[1 + \tanh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) \pm i\,\text{sech}\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{28}$$

From reference (Xie and Tang, 2004), we are told that the Burgers equation (1) admits the following singular solitary wave solution:

$$u_B = -\alpha k\left(1 + \coth\frac{k}{2}\xi\right) \tag{29}$$

and that the KdV equation (2) admits the following singular solitary wave solution:

$$u_K = -12\beta k^2 \,\text{cosech}^2\, k\xi \tag{30}$$

In view of Eq. (29) along with Eq. (30), and making use of Eq. (4) together with Eq. (7), we assume that Eq. (3) has the following ansatz solution:

$$u = a(1 + \coth k\xi) + b \,\mathrm{cosech}^2 k\xi \tag{31}$$

Putting Eq. (31) into Eq. (3), and with the help of the computerized symbolic computation of the powerful Mathematica, we get

$$\begin{aligned} &a^2k - abk - 2bk^2\alpha - 2ak^3\beta - ack \\ &\quad + (a^2k + 2abk - 2b^2k + 2ak^2\alpha - 16bk^3\beta - 2bck)\coth k\xi \\ &\quad + (-a^2k + 4abk + 8bk^2\alpha + 8ak^3\beta + ack)\coth^2 k\xi \\ &\quad + (-a^2k - 2abk + 4b^2k - 2ak^2\alpha + 40bk^3\beta + 2bck)\coth^3 k\xi \\ &\quad + (-3abk - 6bk^2\alpha - 6ak^3\beta)\coth^4 k\xi + (-2b^2k - 24bk^3\beta)\coth^5 k\xi = 0 \end{aligned} \tag{32}$$

Setting each of the coefficients of $\coth^j k\xi$ $(j = 0, 1, 2, \ldots, 5)$ in Eq. (32) to zero leads to a system of over-determined algebraic equations with respect to the unknown variables a and b as follows:

$$a^2k - abk - 2bk^2\alpha - 2ak^3\beta - ack = 0 \tag{33}$$

$$a^2k + 2abk - 2b^2k + 2ak^2\alpha - 16bk^3\beta - 2bck = 0 \tag{34}$$

$$-a^2k + 4abk + 8bk^2\alpha + 8ak^3\beta + ack = 0 \tag{35}$$

$$-a^2k - 2abk + 4b^2k - 2ak^2\alpha + 40bk^3\beta + 2bck = 0 \tag{36}$$

$$-3abk - 6bk^2\alpha - 6ak^3\beta = 0 \tag{37}$$

$$-2b^2k - 24bk^3\beta = 0 \tag{38}$$

Solving the above set of algebraic equations by virtue of Mathematica, we have the following results:
Case 3

$$a = \frac{6\alpha^2}{25\beta}, \qquad b = -\frac{3\alpha^2}{25\beta}, \qquad k = -\frac{\alpha}{10\beta}, \qquad c = \frac{6\alpha^2}{25\beta} \tag{39}$$

Inserting Eq. (39) into Eq. (31) and making use of Eq. (7), we possess the following singular solitary wave solution to the KdV-Burgers equation (3):

$$u_{11} = \frac{12\alpha^2}{25\beta} - \frac{3\alpha^2}{25\beta}\left[1 + \coth\frac{\alpha}{10\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{40}$$

Case 4

$$a = -\frac{6\alpha^2}{25\beta}, \qquad b = -\frac{3\alpha^2}{25\beta}, \qquad k = \frac{\alpha}{10\beta}, \qquad c = -\frac{6\alpha^2}{25\beta} \tag{41}$$

Substituting Eq. (41) into Eq. (31) and taking into account Eq. (7), we obtain the following singular solitary wave solution of the KdV-Burgers equation (3):

$$u_{12} = -\frac{3\alpha^2}{25\beta}\left[1 + \coth\frac{\alpha}{10\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)\right]^2 \tag{42}$$

Making use of the foregoing identity (21) and the following identity:

$$\coth\frac{x}{2} = \frac{\sinh x}{\cosh x - 1} \tag{43}$$

then Eq. (40) and (42) can be reduced respectively to,

$$u_{13} = \frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\frac{\sinh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)}{\cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) - 1} - \frac{6\alpha^2}{25\beta}\frac{1}{\cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) - 1} \tag{44}$$

$$u_{14} = -\frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\frac{\sinh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)}{\cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) - 1} - \frac{6\alpha^2}{25\beta}\frac{1}{\cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) - 1} \tag{45}$$

Similarly, making use of the identity (21) and the following identity:

$$\coth\frac{x}{2} = \frac{\cosh x + 1}{\sinh x} \tag{46}$$

Eq. (40) and (42) can be converted respectively to,

$$u_{15} = \frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\frac{\cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) + 1}{\sinh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right)} - \frac{6\alpha^2}{25\beta}\frac{1}{\cosh\frac{\alpha}{5\beta}\left(x - \frac{6\alpha^2}{25\beta}t\right) - 1} \tag{47}$$

$$u_{16} = -\frac{6\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\frac{\cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) + 1}{\sinh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right)} - \frac{6\alpha^2}{25\beta}\frac{1}{\cosh\frac{\alpha}{5\beta}\left(x + \frac{6\alpha^2}{25\beta}t\right) - 1} \tag{48}$$

Apparently, the solutions u_1, u_2, and u_{11} are the same as those given in references (Feng, 2002; Fu *et al.*, 2004). In the meantime, the solution u_{12} is similar to that obtained in reference (Zhang and Zhang, 2000), and the solutions u_3, u_{13} are similar to those given in reference (Guo and Zhang, 2002) as well. The rest of solutions, however, are the new ones to the KdV-Burgers equation which can not be seen in literature to our knowledge.

Finally, it should be remarked that we have verified each solution (especially each new one) by putting them into the original equation utilizing Mathematica.

3. CONCLUSIONS

In summary, by analyzing carefully the features of the Burgers equation and KdV equation as well as KdV-Burgers equation, we put forward a superposition method to construct the solitary wave solutions of the KdV-Burgers equation from those of the Burgers equation and KdV equation. Many explicit and exact solitary wave solutions to the KdV-Burgers equation, which include some new ones, are deduced by means of this approach. We believe that the technique used herein may be applied to find the solitary wave solutions of other nonlinear wave equations which are of the similar features stated above such as the KdV-Burgers-Kuramoto equation (e.g., we may construct its solutions from those of the KdV equation and Kuramoto–Sivashinsk equation); we leave this to future study.

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REFERENCES

Duffy, B. R. and Parkes, E. J. (1996). *Physics Letters A* **214**, 271.
Feng, Z. (2002). *Journal of Physics A: Mathematical and General* **35**, 343.
Fu, Z. T., *et al*. (2001). *Physics Letters A* **290**, 72.
Fu, Z. T., *et al*. (2004). *Communication in Theoretical Physics* **41**, 527 (Beijing, China).
Guo, G. P. and Zhang, J. F. (2002). *Acta Physica Sinica* **51**, 1159 (in Chinese).
Liu, C. P. (2003). *Physics Letters A* **312**, 41.
Liu, S. D. and Liu, S. K. (1991). *Communication in Theoretical Physics* **16**, 497 (Beijing, China).
Liu, S. K., *et al*. (2001). *Physics Letters A* **289**, 69.
Liu, S. K., *et al*. (2001). *Applied Mathematics and Mechanics* **22**, 326 (in Chinese).
Malfliet, W. (1992). *American Journal of Physics* **60**, 650.
Peng, Y. Z. (2003). *International Journal of Theoretical Physics* **42**, 863.
Otwinowski, M., *et al*. (1988). *Physics Letters A* **128**, 483.
Parkes, E. J. and Duffy, B. R. (1997). *Physics Letters A* **229**, 217.
Parkes, E. J., *et al*. (2002). *Physics Letters A* **295**, 280.
Wang, M. L. (1995). *Physics Letters A* **199**, 169.

Wang, M. L. (1996). *Physics Letters A* **216**, 67.
Xie, Y. X. and Tang, J. S. (2004). *Acta Physica Sinica* **53**, 2828 (in Chinese).
Yan, Z. Y. and Zhang, H. Q. (1999). *Acta Physica Sinica* **48,** 1962 (in Chinese).
Zhang, J. F. (1998). *International Journal of Theoretical Physics* **38**, 1829.
Zhang, Y. F. and Zhang, H. Q. (2000). *Applied Mathematics and Mechanics* **21**, 1009.